APPROXIMATION IN POLICY SPACE, LINEAR AND NONLINEAR PROGRAMMING

Richard Bellman

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1700 MAIN ST. · SANTA MONICA · CALIFORNIA

SUMMARY

In recent papers we have indicated the applicability of the technique of successive approximations to a variety of nonlinear and multi-dimensional problems arising in the theory of dynamic programming. Here we indicate how the method, in the guise of approximation in policy space, can be used to yield monotone approximation for linear, quadratic and non-linear programming. Problems of actual convergence will be discussed elsewhere.

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1. Introduction

The purpose of this paper is to present some applications of the technique of successive approximations, the general factotum of analysis, to some of the basic problems of linear, quadratic and general nonlinear programming.

We shall not discuss any of the interesting and important problems of convergence that arise in this way, but will content ourselves with the quite simple proof of monotone approximation.

The simplicity of proof is a consequence of the fact that we are using a basic concept of dynamic programming, [1], "approximation in policy space."

2. Successive Approximation and Linear Programming

Consider the problem of maximizing the linear form

$$(1) \qquad \sum_{i=1}^{N} b_i x_i ,$$

where the x_i are subject to the restrictions

$$(2) \quad (a) \quad x_i \geq 0, \quad i = 1, 2, \ldots, N,$$

$$(b) \quad \sum_{j=1}^{N} a_{ij} x_j \leq c_i, \quad i = 1, 2, \ldots, M.$$

We suppose that $a_{ij} \geq 0$, $c_i \geq 0$, with a sufficient number of

the a_{ij} positive so that the problem is sensible.

Let $x^0 = (x_1^0, x_2^0, \ldots, x_N^0)$ be any set of x_i satisfying the constraints in (2a) and (2b), and let

(3) $$x = (x_1^0, x_2^0, \ldots, x_{N-2}^0, x_{N-1}, x_N).$$

We now proceed to maximize the linear expression

(4) $$b_{N-1}x_{N-1} + b_N x_N$$

subject to the constraints

(5) (a) $x_{N-1},\ x_N \geq 0,$

(b) $$\sum_{j=1}^{N-2} a_{ij}x_j^0 + \sum_{j=N-1}^{N} a_{ij}x_j \leq c_i, \quad i = 1, 2, \ldots, M.$$

Let values of x_{N-1}, x_N determined by this readily resolved problem be x_{N-1}^1, x_N^1. Denote the vector $(x_1^0, x_2^0, \ldots, x_{N-2}^0, x_{N-1}^1, x_N^1)$ by x^1 and write

(6) $$x^1 = (x_1^1, x_2^1, \ldots, x_N^1).$$

Now fix the values of $x_2^1, \ldots, x_{N-1}^1$, and consider the new vector x given by

(7) $$x = (x_1, x_2^1, \ldots, x_{N-1}^1, x_N).$$

Consider the problem of maximizing the linear expression

(8) $$b_1x_1 + b_N x_N$$

subject to the constraints

(9) (a) $x_1,\ x_N \geq 0,$

(b) $\sum_{j=2}^{N-1} a_{ij}x_j + \sum_{j=1,N} a_{ij}x_j \leq c_i, \quad i = 1, 2, \ldots, M.$

This again is simply resolved. Call a set of maximizing values $x_1^2,\ x_N^2,$ and write

(10) $$x^2 = (x_1^2, x_2^1, \ldots, x_{N-1}^1, x_N^2) = (x_1^2, x_2^2, \ldots, x_N^2).$$

The next step is to fix $x_3^2,\ x_4^2,\ \ldots,\ x_N^2$ and maximize over $x_3,\ x_4$. Continuing in this way, we obtain a sequence of successive approximations to the true maximum.

3. Monotonicity of Approximation

To show that we obtain a sequence of vectors $\{x^n\}$ which yield a monotone increasing sequence of values for $\sum_{i=1}^{N} b_i x_i$, we proceed as follows.

It is clear that having chosen x^0, we can always choose the last two components of x in (2.3) to be x_{N-1}^0 and x_N^0. Consequently, when we maximize over x_{N-1} and x_N, we automatically obtain a value of $\sum_{i=1}^{N} b_i x_i$ at least as large as that given by $\sum_{i=1}^{N} b_i x_i^0$.

4. Nonlinear Programming

It is clear that the same technique can be applied to the problem of maximizing $\sum_{i=1}^{N} b_i(x_i)$ subject to a series of inequalities of the form

$$(1)\quad (a)\quad x_i \geq 0,$$

$$(b)\quad \sum_{j=1}^{N} a_j(x_j) \leq c_i, \quad i = 1, 2, \ldots, M.$$

A difference is that the problem of maximizing over two variables in general will require a computational solution and will not possess a simple explicit solution as in the linear case.

5. Convergence

It is not at all clear, even in the linear case where there is a unique maximum, rather than a set of local maxima, as may occur in the nonlinear case, that the sequence of values converges to the actual maximum. That the sequence converges is clear, but it is not clear to what it converges.

Consequently, if the sequence of values sticks at a particular value, the thing to do is to upset the cyclic arrangement described above and to study other sets of two values at a time. Thus, instead of (1,2), (2,3), ..., (N-1,N), (N,1), we can use (1,4), (4,7), ..., and so on.

Furthermore, instead of using a fixed sequence of pairs which increases the probability of pathological behavior, we can use a random selection of pairs.

It would be interesting to know, if the simple technique described above does not yield convergence, whether this is the usual situation, or whether it occurs with small probability.

6. Quadratic Programming

As another application of successive approximations, let us

consider the problem of maximizing (x,Ax), where A is a positive definite matrix, subject to the constraints of (2.2).

Write

$$(1) \qquad (x,Ax) = (y + x - y,\ A(y + x - y))$$

$$= (y,\ Ay) + 2(x - y,\ Ay) + (x - y,\ A(x - y)).$$

It follows that

$$(2) \qquad (x,Ax) \geq (y,Ay) + 2(x - y,\ Ay)$$

for all y and x.

Consider then, in place of the original nonlinear problem, the problem of maximizing

$$(3) \qquad J(x,x^0) = (x^0,Ax^0) + 2(x - x^0,Ax^0)$$

over all x subject to the constraints of (2.2), where x^0 is an initial guess satisfying (2.2).

Since $x = x^0$ is a feasible choice, it follows that any x which yields the maximum of (3) furnishes a value, x^1, which yields a larger value of (x,Ax). For

$$(4) \qquad (x^1,Ax^1) \geq (x^0,Ax^0) + 2(x^1 - x^0,Ax^0) \geq (x^0,Ax^0),$$

by virtue of the maximum property.

The original nonlinear problem has thus been reduced to a sequence of linear problems. We shall discuss the convergence question elsewhere.

REFERENCES

1. Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, New Jersey, 1957.